Contact symmetries and solutions by reduction of partial differential equations

1994 J. Phys. A: Math. Gen. 27 177

(http://iopscience.iop.org/0305-4470/27/1/011)

J. Phys. A: Math. Gen. 27 (1994) 177–184. Printed in the UK

Contact symmetries and solutions by reduction of partial differential equations

Edvige Pucci and Giuseppe Saccomandi
Istituto di Energetica, Universitá degli Studi di Perugia, 06100, Perugia, Italy

Received 29 March 1993, in final form 16 September 1993

Abstract. The correspondence between a contact symmetry of a second-order PDE $\mathscr{E}$ and a point symmetry of the equivalent first order system $\mathscr{S}$ is used to determine a class of solutions for $\mathscr{E}$ which may not always be invariant under contact transformation. We name these solutions *pseudo-invariant* because they are determined from a point symmetry of $\mathscr{S}$ via a reduction method. Invariant solutions under the related contact transformation are included in the family of *pseudo-invariant* solutions.

1. Introduction

The application of Lie group theory to obtain exact analytical solutions (the so-called *invariant solutions*) of a partial differential equation (PDE) $\mathscr{E}$ is widely known (Bluman and Kumei 1989, Olver 1986). A group of point transformations (i.e. a group with infinitesimal generators depending only on the variables and the unknown function) can be used to reduce the number of variables of $\mathscr{E}$ by means of a well-defined reduction algorithm. In particular, if $\mathscr{E}$ is a PDE in two variables, in such a way we obtain an ordinary differential equation.

In the following we will narrow our attention to the PDE of the second order in two variables.

If we consider more general transformations as the Lie–Bäcklund (LB) (i.e. with infinitesimal generators which depend also on the derivatives of the unknown function) only in a few particular cases can we obtain the invariant solutions using a reduction procedure. This is possible, for example, for evolution equations, or in the case of LB transformation of the first order (i.e. with infinitesimal generators which depend on the first derivatives), by using the characteristic system associated with the invariant surface conditions which is again a first order PDE, but no longer quasilinear (Bluman and Kumei 1989).

As is well known, first-order LB transformations are equivalent to contact transformations. For the PDE of the second order they are also equivalent to the point symmetries of the $\mathscr{S}$ system generated from $\mathscr{E}$ by introducing the first derivatives of the unknown function as new auxiliary unknowns.

The aim of this paper is to show that the point symmetries of $\mathscr{S}$ can be used to obtain *not only the solutions of* $\mathscr{E}$ *invariant under the corresponding contact symmetries*, but also a wider class of solutions which are not necessarily invariant. We will call this class of solutions *pseudo-invariant solutions* since they can be obtained by a reduction method as the invariant solutions.

0305-4470/94/010177+08 $07.50

We will call $\mathcal{F}^*$ the family of the solutions of the invariant surfaces conditions of the point symmetries of $\mathcal{S}$ and $\mathcal{F}$ the family of the solutions of the invariant surface condition of the corresponding contact symmetry of $\mathscr{E}$. The family $\mathcal{F}^*$ is larger than the family $\mathcal{F}$. This is due to the fact that in $\mathcal{F}^*$ we do not link the auxiliary unknowns to the principal unknown function. For this reason, it is possible to find solutions of $\mathscr{E}$ in $\mathcal{F}^*$ which are not enclosed in $\mathcal{F}$.

The algorithm to find pseudo-invariant solutions is well defined. Once we know the $\mathcal{F}^*$ family, the direct introduction of the unknown function and its derivatives in $\mathscr{E}$ generates a relation between the similarity variable and the similarity functions where one of the two old variables appears as a parameter. By imposing that this relation be true for each value of the parameter, we obtain a system of ODEs in the similarity functions. The solutions of this last system, introduced in $\mathcal{F}^*$, determine the pseudo-invariant solutions. Obviously, invariant solutions are a subclass of the pseudo-invariant solutions.

Examples of pseudo-invariant solutions are given for a family of evolution equation and for a family of Monge–Ampère equations.

2. Invariant and pseudo-invariant solutions

A one-parameter Lie group of contact transformations is characterized by the operator

$$\mathscr{X}=\xi(x,t,u,u_x,u_t)\frac{\partial}{\partial x}+\tau(x,t,u,u_x,u_t)\frac{\partial}{\partial t}+\eta(x,t,u,u_x,u_t)\frac{\partial}{\partial u} \tag{2.1}$$

provided that the contact conditions

$$\frac{\partial\eta}{\partial u_x}-\frac{\partial\xi}{\partial u_x}u_x-\frac{\partial\tau}{\partial u_x}u_t=0 \qquad \frac{\partial\eta}{\partial u_t}-\frac{\partial\xi}{\partial u_t}u_x-\frac{\partial\tau}{\partial u_t}u_t=0 \tag{2.2}$$

are preserved.

A contact symmetry is admitted by a second-order PDE ($\mathscr{E}$)

$$\Delta(x,t,u,u_x,u_t,u_{xx},u_{xt},u_{tt})=0 \tag{2.3}$$

where Δ is a smooth function, iff

$$\mathscr{P}_2\Delta|_{\Delta=0}=0 \tag{2.4}$$

where $\mathscr{P}_2$ is the second extended infinitesimal operator of (2.1) (Olver 1986).

The first-order system $\mathcal{S}$ equivalent to $\mathscr{E}$ is

$$u_x=v \qquad u_t=w \qquad v_t-w_x=0 \tag{2.5a}$$

$$\hat{\Delta}(x,t,u,v,w,v_x,v_t,w_t)=0. \tag{2.5b}$$

A point transformation, admitted by $\mathcal{S}$, defined by the infinitesimal operator

$$\hat{\mathscr{X}}=\hat{\xi}\frac{\partial}{\partial x}+\hat{\tau}\frac{\partial}{\partial t}+\hat{\eta}\frac{\partial}{\partial u}+\hat{\phi}\frac{\partial}{\partial v}+\hat{\psi}\frac{\partial}{\partial w} \tag{2.6}$$

where the infinitesimal generators $\hat{\xi}$, $\hat{\tau}$, $\hat{\eta}$, $\hat{\phi}$ on $\hat{\psi}$, depend on x, t, u, v, w, is a point symmetry of $\mathcal{S}$ iff

$$\hat{\mathscr{P}}_1 S|_{S=0}=0. \tag{2.7}$$

where $\hat{\mathscr{P}}_1$ is the first extended infinitesimal operator of (2.6).

The operator $\hat{\mathscr{P}}_1$, applied to (2.5a) and (2.5b), gives the following relations

$$\hat{\eta}_w = v\hat{\xi}_w + w\hat{\tau}_w \qquad \hat{\eta}_v = v\hat{\xi}_v + w\hat{\tau}_v, \tag{2.8}$$

$$\hat{\phi} = \hat{\eta}_x + \hat{\eta}_u v - v(\hat{\xi}_x + \hat{\xi}_u v) - w(\hat{\tau}_x + \hat{\tau}_u v) \tag{2.9}$$

$$\hat{\psi} = \hat{\eta}_t + \hat{\eta}_u w - v(\hat{\xi}_t + \hat{\xi}_u w) - w(\hat{\tau}_t + \hat{\tau}_u w). \tag{2.10}$$

From (2.8), by derivation, we have $\hat{\xi}_w = \hat{\tau}_v$ then

$$\hat{\mathscr{P}}_1(v_t - w_x)|_{S=0} \equiv 0.$$

The equivalence between (2.1), (2.2) and (2,6), (2.8) is obvious, and (2.9) and (2.10) are the first extensions of η. For these reasons we have the equivalence between (2.4) and

$$\hat{\mathscr{P}}_1 \hat{\Delta}|_{S=0} = 0$$

and also a one-to-one correspondence between the contact symmetries of $\mathscr{E}$ and the point symmetries of $\mathscr{S}$ (Olver 1979).

In the following, we will consider only proper contact symmetries, i.e. symmetries which are not equivalent to the $\mathscr{E}$ point symmetries.

The solutions of $\mathscr{E}$ invariant under a contact symmetry are the solutions of $\mathscr{E}$ which verify the invariant surface condition

$$F(x,t,u,u_x,u_t) \equiv \xi(x,t,u,u_x,u_t)u_x + \tau(x,t,u,u_x,u_t)u_t - \eta(x,t,u,u_x,u_t) = 0 \tag{2.11}$$

which is a non-quasilinear first-order PDE with characteristic system

$$\frac{dx}{F_{u_x}} = \frac{dt}{F_{u_t}} = \frac{du}{u_x F_{u_x} + u_t F_{u_t}} = -\frac{du_x}{F_x + F_u u_x} = -\frac{du_t}{F_t + F_u u_t}. \tag{2.12}$$

Every solution of (2.12), verifying (2.11), is a characteristic strip to which a characteristic curve is associated.

The characteristic strips are then defined by $F=0$ and by three integrals of (2.12)

$$h_i(x,t,u,u_x,u_t) = \gamma_i \qquad i = 0,1,2 \tag{2.13}$$

with $\partial(F,h_1,h_2,h_3)/\partial(u,u_x,u_t)$ of rank 3 (Courant and Hilbert 1962).

Every solution of (2.11) is a one-parameter family of characteristic curves. Let $\gamma_0 = \zeta$ be the parameter (similarity variable) and solving for u, u_x, and u_t the $F=0$ and the remaining (2.13), where we have set $\gamma_1 = K_1(\zeta)$ and $\gamma_2 = K_2(\zeta)$, we get

$$u = U(x,t,\zeta,K_1(\zeta),K_2(\zeta)) \tag{2.14}$$

$$u_x = U_x(x,t,\zeta,K_1(\zeta),K_2(\zeta)) \tag{2.15}$$

$$u_t = U_t(x,t,\zeta,K_1(\zeta),K_2(\zeta)) \tag{2.16}$$

$$G(x,t,\zeta,K_1(\zeta),K_2(\zeta)) = 0. \tag{2.17}$$

Equation (2.14), with ζ defined implicitly by (2.17) as a function of x and t, is the $\mathscr{F}$-family of solutions of (2.11). The invariant solutions of $\mathscr{E}$ under the contact transformation are the functions in $\mathscr{F}$ solutions of (2.3)..

Related to every contact symmetry (2.1) of $\mathscr{E}$, we can define the point symmetry of $\mathscr{S}$ with invariant surface conditions

$$\hat{\xi}u_x + \hat{\tau}u_t - \hat{\eta} = 0 \qquad \hat{\xi}v_x + \tau v_t - \hat{\phi} = 0 \qquad \hat{\xi}w_x + \hat{\tau}w_t - \hat{\psi} = 0 \tag{2.18}$$

and characteristic system

$$\frac{\mathrm{d}x}{\hat{\xi}}=\frac{\mathrm{d}t}{\hat{\tau}}=\frac{\mathrm{d}u}{\hat{\eta}}=\frac{\mathrm{d}v}{\hat{\phi}}=\frac{\mathrm{d}w}{\hat{\psi}}. \tag{2.19}$$

We remark that in (2.18) and (2.19) there is no link between u, v and w.

The solutions of the equations (2.19) are defined by four integrals

$$k_i(x,t,u,v,w)=c_i \qquad i=0,1,2,3 \tag{2.20}$$

where $\partial(k_0,k_1,k_2,k_3)/\partial(u,v,w)$ is of rank 3. The relation

$$\hat{F}\equiv\hat{\xi}v+\hat{\tau}w-\hat{\eta}=0 \tag{2.21}$$

in an invariant relation (Levi Civita 1906) for the differential system (2.19); in fact by taking into account (2.8), (2.9) and (2.10), we have

$$\mathscr{X}\hat{F}\equiv(\hat{\eta}_u-\hat{\xi}_u v-\hat{\tau}_u w)\hat{F}. \tag{2.22}$$

It is possible to obtain (2.21) from (2.11) via the formal substitution $u_x\to v$ and $u_t\to w$. Then, when (2.11) is verified, (2.12) is only formally different from (2.19) and this means that the family of the characteristic strips coincides with the family of the solutions of (2.19) that satisfies (2.21).

This last family is defined by $\hat{F}=0$ and by

$$\hat{h}_i(x,t,u,v,w)=\gamma_i \qquad i=0,1,2$$

obtained from (2.13) via the formal substitution $(u_x\to v,\ u_t\to w)$.

The functions u, v and w solutions of (2.18) can be obtained from (2.20) as a one-parameter family of characteristic curves. Choosing $c_0=z$ as similarity variable and $c_i=H_i(z)$ as similarity functions, if we solve for u, v and w the remaining relations in (2.20), we obtain the solution of (2.19) as

$$u=U(x,t,z,H_1(z),H_2(z),H_3(z)) \tag{2.23}$$

$$v=V(x,t,z,H_1(z),H_2(z),H_3(z)) \tag{2.24}$$

$$w=W(x,t,z,H_1(z),H_2(z),H_3(z)) \tag{2.25}$$

$$\Gamma(x,t,z,H_1(z),H_2(z),H_3(z))=0. \tag{2.26}$$

Equation (2.23), where z is implicitly defined by (2.26), are the $\mathscr{F}^*$ family. The invariant solutions of $\mathscr{S}$ are the functions (2.23), . . . , (2.26) that verify the $\mathscr{S}$ system.

By substitution in $\mathscr{S}$ of these functions we obtain a system Σ of ordinary differential equations in the $H_i(z)$ unknown functions. Introducing the solution of Σ in (2.23) we obtain all the invariant solutions of $\mathscr{E}$ under the contact transformation, since the relation (2.21) is verified.

The pseudo-invariant solutions of $\mathscr{E}$ are the solutions of $\mathscr{E}$ in $\mathscr{F}^*$; they do not always satisfy (2.21). The direct substitution of (2.23), where z is defined by (2.26), in (2.3) generates a differential relation between the $H_i(z)$ functions where one of the two variables x and t must be considered as a parameter. Indeed it is always possible to solve (2.26) for x or t.

By imposing that the above relation holds true for every value of the parameter, we obtain a differential system Σ^* where the unknown functions are always $H_i(z)$. Every solution of Σ^*, introduced in (2.23), gives rise to a pseudo-invariant solution of (2.3). The solutions of Σ^* which are also solutions of Σ are obviously the invariant solutions.

In the next section, we will give non-trivial examples of pseudo-invariant solutions.

3. Examples

We consider the equation

$$u_{tt}(u_{xx}-\lambda)-u_{xt}^2=0 \tag{3.1}$$

where λ is an arbitrary parameter. The equivalent $\mathcal{S}$ system

$$u_x=v \qquad u_t=w \qquad v_t=w_x \qquad w_t(v_x-\lambda)-v_t^2=0 \tag{3.2}$$

is invariant under the point symmetry with infinitesimal generators

$$\hat{\xi}=v \qquad \hat{\tau}=\lambda t \qquad \hat{\eta}=\lambda u+\frac{v^2}{2} \qquad \hat{\phi}=\lambda v \qquad \hat{\psi}=0. \tag{3.3}$$

The invariant surface conditions are

$$\begin{aligned} &vu_x+\lambda tu_t-\lambda u-v^2/2=0 \\ &vv_x+\lambda tv_t-\lambda v=0 \\ &vw_x+\lambda tw_t=0 \end{aligned} \tag{3.4}$$

and so the integrals of the corresponding characteristic system (2.19) are

$$\frac{\lambda t}{v}=z \qquad w=H_1(z) \qquad \lambda x-v=H_2(z) \qquad \frac{u}{v}-\frac{x}{2}=H_3(z). \tag{3.5}$$

Then the solutions of (3.4) are

$$u=(\lambda x-H_2(z))\left(\frac{x}{2}+H_3(z)\right) \qquad v=\lambda x-H_2(z) \qquad w=H_1(z) \tag{3.6}$$

where z is defined by

$$\lambda t-z(\lambda x-H_2(z))=0. \tag{3.7}$$

The substitution of (3.6) in $\mathcal{S}$ gives

$$zH_1'-H_2'=0 \qquad 2(\lambda H_3-zH_1)+H_2=0$$

where $H_i'=\mathrm{d}H_i/\mathrm{d}z$. In doing so we obtain

$$H_1=\frac{H_2}{z}+\int\frac{H_2}{z^2}\,\mathrm{d}z+f_0 \tag{3.8a}$$

$$H_3=\frac{1}{2\lambda}\left[H_2+2z\int\frac{H_2}{z^2}\,\mathrm{d}z+2f_0z\right] \tag{3.8b}$$

where $H_2(z)$ is an arbitrary function and f_0 an arbitrary constant. Then the invariant solutions of $\mathcal{S}$ are

$$u=\frac{t}{2z}\left(\lambda x+H_2+2z\int\frac{H_2}{z^2}\,\mathrm{d}z+2f_0z\right) \tag{3.9}$$

$$v=\lambda x-H_2 \qquad w=\frac{H_2}{z}+\int\frac{H_2}{z^2}\,\mathrm{d}z+f_0 \tag{3.10}$$

where z is defined by (3.7).

The (3.9) are the solutions of $\mathscr{E}$, invariant under the contact symmetry, defined by

$$\xi = u_x \qquad \tau = \lambda t \qquad \eta = \lambda u + \frac{u_x^2}{2} \tag{3.11}$$

with invariant surface condition

$$\lambda t u_t + \frac{u_x^2}{2} - \lambda u = 0. \tag{3.12}$$

Now we will find the pseudo-invariant solutions of $\mathscr{E}$. By substitution of $(3.6)_1$ in (3.1) we obtain that the system Σ^*, in this case, is given by the single equation

$$H_2'(2\lambda z H_3' - 2\lambda H_3 - H_2' z - H_2) = 0.$$

Two cases are possible:

(a) $H_2 = k$ (k arbitrary constant).
Then we have

$$u = (\lambda x - k)\left[H_3\left(\frac{\lambda t}{\lambda x - k}\right) + \frac{x}{2}\right] \tag{3.13}$$

where H_3 is an arbitrary function.

(b) $2\lambda z H_3' - 2\lambda H_3 - H_2' z - H_2 = 0$.
Then H_3 is defined by (3.8*b*) and the function u is the same as (3.9).

The pseudo-invariant solutions in the case of (a) are invariant under (3.11) iff $H_3 = \lambda t/(\lambda x - k) - k/2\lambda$; in the case (b) all the pseudo-invariant solutions are also invariant solutions.

For $\lambda = 0$ the (3.1) is the Monge–Ampère equation. The invariant solutions of $\mathscr{S}_0$ (the system obtained from (3.2) when $\lambda = 0$) are $u = H(t)$, $v = 0$, $w = H'(t)$, where H is an arbitrary function. On the other hand, the pseudo-invariant solutions are defined by

$$u = \frac{kx}{2} + H(t)$$

where H is an arbitrary function (more on contact symmetries for the Monge–Ampère equation can be found in Ibragimov 1985).

We consider the family of evolution equations

$$u_t = \frac{\gamma'}{\gamma^2}(\gamma x - 1)u_x + R\left(\frac{1}{u_x} - \frac{\gamma}{u_{xx}}\right) \tag{3.14}$$

where $\gamma(t)$ and $R((1/u_x) - (\gamma/u_{xx}))$ are smooth arbitrary functions.

In this case, the equivalent $\mathscr{S}$ system

$$u_x = v \qquad u_t = w \qquad v_t = w_x \qquad w = \frac{\gamma'}{\gamma^2}(\gamma x - 1)v + R\left(\frac{1}{v} - \frac{\gamma}{v_x}\right) \tag{3.15}$$

is invariant under the point symmetry with infinitesimal generators

$$\xi = \frac{1}{v} \qquad \tau = 0 \qquad \eta = 1 + \gamma x - \ln(v) \qquad \phi = \gamma \qquad \psi = \gamma' x.$$

From the characteristic system we obtain t as a similarity variable and u, v and w defined by

$$u=\frac{1}{\gamma}(H_1+(H_2+1)\exp(\gamma x-H_2)) \tag{3.16a}$$

$$v=\exp(\gamma x-H_2) \tag{3.16b}$$

$$w=-\frac{H_3}{\gamma}+\frac{\gamma'}{\gamma^2}(\gamma x-1)\exp(\gamma x-H_2) \tag{3.16c}$$

where now H_1, H_2 and H_3 are arbitrary functions of t.

The system Σ is

$$\gamma H_1'-\gamma' H_1-\gamma^2 R(0)=0 \qquad H_2=0 \qquad H_3+\gamma R(0)=0$$

then $H_1=\gamma t R(0)+c_0\gamma$, $H_2=0$, $H_3=-\gamma R(0)$ where c_0 is an arbitrary constant.

The invariant solutions of $\mathscr{E}$ under the correspondent contact symmetries are

$$u=\frac{1}{\gamma}(\gamma t R(0)+c_0\gamma+\exp(\gamma x)). \tag{3.17}$$

The pseudo-invariant solutions are obtained considering the relation found by direct introduction of (3.16*a*) in (3.14). Requiring that this relation holds for every value of the x parameter, we have the Σ^* system of equations

$$\gamma H_1'-\gamma' H_1-\gamma^2 R(0)=0 \qquad H_2'=0.$$

Then the pseudo-invariant solutions are

$$u=\frac{1}{\gamma}(\gamma t R(0)+c_0\gamma+c_1\exp(\gamma x))$$

where c_1 is an arbitrary constant.

4. Concluding remarks

We have examined thoroughly the possibility to compute exact solutions of the equation $\mathscr{E}$ starting from the point symmetries (PS) of the system $\mathscr{S}$ corresponding to contact symmetries of $\mathscr{E}$. The analysis of the characteristic systems (2.12) and (2.19) has suggested the main result: the solutions of $\mathscr{E}$ with functional form defined by the the invariant surface conditions of PS, i.e. the pseudo-invariant, are a wide class than the ones which satisfy (2.11), i.e. the invariant solutions. Then more exact solutions by reduction can be found in such a way.

A priori there is no relationship between this method and other methods of reduction related to group theory, as non-classical and weak symmetries (Bluman and Cole 1969, Pucci and Saccomandi 1992), or to direct methods (Clarkson and Kruskal 1989, Rubel 1991, Galaktionov 1990).

On the other hand, after the determination of the pseudo-invariant solutions, it is easy to check if, by chance, these can be found also via other procedures. For example the solutions (3.13) of the equation (3.1) are invariant under the non-classical symmetry with generators

$$\xi=\frac{\lambda x-k}{\lambda t} \qquad \tau=1 \qquad \eta=\frac{(\lambda x-k)^2+2\lambda u}{2\lambda t}.$$

Then, by using the results in Pucci (1992), we know that the solutions (3.13) can also be obtained by the method of Clarkson and Kruskal; moreover, when $H_3 = \phi(t)\psi(x)$, it is easy to obtain them via the nonlinear separation method (Galaktionov 1990). However, generally, nothing can be said on the relationship between our method and these other procedures.

We point out that to find the pseudo-invariant solutions of $\mathscr{E}$ we need only to compute the PS of $\mathscr{S}$, which are characterized by a linear system of determining equations; this is no more the case for non-classical or weak symmetries that, also in the simplest cases, are characterized by highly nonlinear systems. Nonlinear equations must be solved also when direct methods are applied and moreover such methods are not entirely algorithmic.

A procedure to obtain new exact solutions by reduction, which is similar to the one of this paper, has been successfully applied to invariant PDE under potential symmetries (Pucci and Saccomandi 1993).

References

Bluman G W and Cole J D 1969 *J. Math. Mech.* **18** 1025
Bluman G W and Kumei S 1989 *Symmetries and Differential Equations, Applied Math. Sci.* **81** (New York: Springer)
Clarkson P A and Kruskal M D 1989 *J. Math. Phys.* **30** 2201
Courant R and Hilbert D 1962 *Methods of Mathematical Physics* vol II (New York: Interscience)
Galaktionov V A 1990 *Diff. Int. Eqs.* **3** 863
Ibragimov M H 1985 *Transformation Groups Applied to Mathematical Physics* MIA (Dordrecht: Reidel)
Levi Civita T 1906 *Prace Matematyczno-Fizycznych* **XVII** 1 (also in: Levi Civita T 1961 *Opere* vol II (Bologna: Zanichelli)
Olver P J 1979 *J. Diff. Geom.* **14** 497
Olver P J 1986 *Applications of Lie Groups to Differential Equations* GTM 107 (New York: Springer)
Pucci E 1992 *J. Phys. A: Math. Gen.* **25** 2631
Pucci E and Saccomandi G 1992 *J. Math. Anal. Appl.* **163** 588
Pucci E and Saccomandi G 1993 *J. Phys. A: math. Gen.* **26** 681
Rubel L A 1991 *Illinois J. Math.* **35** 690